

# $L^p$ SPHERICAL MULTIPLIERS ON HOMOGENEOUS TREES

DARIO CELOTTO, STEFANO MEDA AND BLAŻEJ WRÓBEL

ABSTRACT. We characterise, for each  $p$  in  $[1, \infty) \setminus \{2\}$ , the class of  $L^p$  spherical multipliers on homogeneous trees in terms of  $L^p$  Fourier multipliers on the torus.

## 1. INTRODUCTION

A homogeneous tree of degree  $q$  is a connected graph  $\mathcal{T}$  with no loops such that any point  $x$  of  $\mathcal{T}$  has exactly  $q + 1$  neighbours. Henceforth we assume that  $q \geq 2$ . We endow  $\mathcal{T}$  with the counting measure and the natural distance.

Fix an arbitrary reference point  $o$  in  $\mathcal{T}$ , denote by  $G$  the group of isometries of  $\mathcal{T}$  and by  $G_o$  the stabiliser of  $o$  in  $G$ . The group  $G_o$  is a maximal compact subgroup of  $G$ . The map  $g \mapsto g \cdot o$  identifies  $\mathcal{T}$  with the coset space  $G/G_o$ . Thus, a function  $f$  on  $\mathcal{T}$  gives rise to a  $G_o$ -invariant function  $f'$  on  $G$  by the formula  $f'(g) = f(g \cdot o)$ , and every  $G_o$ -invariant function arises in this way. The distance of  $x$  from  $o$  will be denoted by  $|x|$ . A function  $f$  on  $\mathcal{T}$  is called radial if  $f(x)$  depends only on  $|x|$ , or equivalently if  $f$  is  $G_o$ -invariant, or if  $f'$  is  $G_o$ -bi-invariant.

It is well known that  $G$ -invariant linear operators on  $L^p(\mathcal{T})$  correspond to bounded linear operators on  $L^p(G/G_o)$  given by convolution on the right with  $G_o$ -bi-invariant kernels. We denote by  $Cv_p(\mathcal{T})$  the space of radial functions on  $\mathcal{T}$  associated to these  $G_o$ -bi-invariant kernels. The norm of an element  $k$  in  $Cv_p(\mathcal{T})$  is then defined as the norm of the corresponding operator on  $L^p(G/G_o)$ , equivalently as the norm of the associated  $G_o$ -invariant operator on  $L^p(\mathcal{T})$ , and it is denoted by  $\|k\|_{Cv_p(\mathcal{T})}$ .

We also denote by  $Cv_p(\mathbb{Z})$  the space of the convolution kernels associated to the translation invariant operators on  $L^p(\mathbb{Z})$ . The norm of a function  $k$  in  $Cv_p(\mathbb{Z})$  is the

---

2010 *Mathematics Subject Classification.* Primary 43A90, 20E08, 43A85.

*Key words and phrases.* spherical multiplier, homogeneous tree, harmonic analysis.

Work partially supported by PRIN 2010 “Real and complex manifolds: geometry, topology and harmonic analysis”. The first two named authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The research of the third named author was carried over while he was *Assegnista di ricerca* at the Università di Milano-Bicocca. He was supported by PRIN 2010 “Real and complex manifolds: geometry, topology and harmonic analysis”, by Polish funds for sciences, National Science Centre (NCN), Poland, Research Project 2014/15/D/ST1/00405, and by Foundation for Polish Science - START scholarship.

$L^p(\mathbb{Z})$  operator norm of the corresponding convolution operator. Set  $\tau := 2\pi/\log q$ , and denote by  $\mathcal{F}$  the Fourier transformation on  $\mathbb{Z}$ , given by

$$(1.1) \quad \mathcal{F}F(s) = \sum_{d \in \mathbb{Z}} F(d) q^{-ids} \quad \forall s \in \mathbb{T},$$

where  $\mathbb{T} = \mathbb{R}/(\tau\mathbb{Z})$ . We denote by  $\mathcal{M}_p(\mathbb{T})$  the space of all (bounded) functions on  $\mathbb{T}$  of the form  $\mathcal{F}k$ , where  $k$  is in  $Cv_p(\mathbb{Z})$ . The norm of a function  $\mathcal{F}k$  in  $\mathcal{M}_p(\mathbb{T})$  is then defined to be the norm of  $k$  in  $Cv_p(\mathbb{Z})$ .

The analogue on trees of a celebrated result of J.L. Clerc and E.M. Stein [CSt] states that if  $k$  is in  $Cv_p(\mathcal{T})$ , then its spherical Fourier transform  $\tilde{k}$  extends to a bounded holomorphic function on the strip  $\mathbf{S}_{\delta(p)}$  (see Section 2 for the definition of  $\mathbf{S}_{\delta(p)}$ ). This necessary condition was sharpened by M. Cowling, Meda and A.G. Setti [CMS2, Theorem 2.1], who proved that if  $k$  is in  $Cv_p(\mathcal{T})$ , then the boundary values  $\tilde{k}_{\delta(p)}$  of  $\tilde{k}$  on the strip  $\mathbf{S}_{\delta(p)}$  belong to  $\mathcal{M}_p(\mathbb{T})$ . They also proved that this condition implies that convolution with  $k$  on the right is a bounded operator on the space of all radial functions in  $L^p(\mathcal{T})$ . The work of these authors was inspired by previous work of by R. Szwarc [Sz] and T. Pytlik [P]. In particular, Pytlik showed that a nonnegative radial function  $k$  is in  $Cv_p(\mathcal{T})$  if and only if  $k$  belongs to the Lorentz space  $L^{p,1}(\mathcal{T})$ . This eventually led Cowling, Meda and Setti to prove a sharp form of the Kunze–Stein phenomenon on the full group  $G$  [CMS2, Theorem 1] (see also [N] for a previous less precise version of this phenomenon), and A. Veca to complement this result by proving an endpoint for  $p = 2$ . We shall prove the following.

**Theorem.** *Suppose that  $p$  is in  $[1, \infty) \setminus \{2\}$ , and that  $k$  is a radial function on  $\mathcal{T}$ . Then  $k$  is in  $Cv_p(\mathcal{T})$  if and only if its spherical Fourier transform  $\tilde{k}$  extends to a Weyl-invariant function on  $\mathbf{S}_{\delta(p)}$  and  $\tilde{k}_{\delta(p)}$  is in  $\mathcal{M}_p(\mathbb{T})$ .*

The proof combines techniques from [CMS2] and a generalisation of a transference result of A.D. Ionescu [I1] for rank one noncompact symmetric spaces.

Our paper is organised as follows. Section 2 provides some background and preliminary results. Section 3 contains a general transference result, which is of independent interest and may be applied to other situations. The main result is proved in Section 4.

We shall use the “variable constant convention”, and denote by  $C$ , possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

## 2. BACKGROUND MATERIAL AND PRELIMINARY RESULTS

We now summarise the main features of spherical harmonic analysis on  $\mathcal{T}$ . Standard references concerning harmonic analysis on trees are the books [FTP, FTN]. Our notation is consistent with that of the papers [CMS1, CMS2, CMS3]. The reader is also referred to

the papers [CS, MS1, MS2, Se1, Se2] for various related aspects of harmonic analysis on homogeneous trees. The spherical functions are the radial eigenfunctions of the standard nearest neighbour Laplacian satisfying the normalisation condition  $\phi(o) = 1$ , and are given by the formula

$$\phi_z(x) = \begin{cases} \left(1 + \frac{q-1}{q+1}|x|\right) q^{-|x|/2} & \forall z \in \tau\mathbb{Z} \\ \left(1 + \frac{q-1}{q+1}|x|\right) q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2 + \tau\mathbb{Z} \\ \mathbf{c}(z) q^{(iz-1/2)|x|} + \mathbf{c}(-z) q^{(-iz-1/2)|x|} & \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}, \end{cases}$$

where  $\tau := 2\pi/\log q$  and  $\mathbf{c}$  is the meromorphic function defined by the rule

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1} \frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}} \quad \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}.$$

It is straightforward to check that for each  $x$  in  $\mathcal{T}$  the function  $z \mapsto \phi_z(x)$  is entire and that

$$|\phi_z(x)| \leq 1 \quad \forall x \in \mathcal{T} \quad \forall z \in \overline{\mathbf{S}}_{1/2}.$$

For each  $p$  in  $[1, \infty]$  we write  $\delta(p)$  for  $|1/p - 1/2|$  and  $p'$  for the conjugate index  $p/(p-1)$ . For any nonnegative real number  $t$ , we denote by  $\mathbf{S}_t$  and  $\overline{\mathbf{S}}_t$  the strip  $\{z \in \mathbb{C} : |\operatorname{Im} z| < t\}$  and its closure, respectively. If  $f$  is a holomorphic function on  $\mathbf{S}_t$ , and  $v$  is in  $(-t, t)$  then  $f_v$  denotes the function on  $\mathbb{R}$  defined by  $f_v(u) = f(u + iv)$ . We also denote by  $f_t$  and  $f_{-t}$  the boundary values of  $f$ , when they exist in the sense of distributions.

The spherical Fourier transform  $\tilde{f}$  of a radial function  $f$  in  $L^1(\mathcal{T})$  is

$$\tilde{f}(z) = \sum_{x \in \mathcal{T}} f(x) \phi_z(x) \quad \forall z \in \overline{\mathbf{S}}_{1/2}.$$

Since the map  $z \mapsto \phi_z$  is even and  $\tau$ -periodic in the strip  $\mathbf{S}_{1/2}$ , so is the function  $\tilde{f}$ . We say that a holomorphic function in a strip  $\mathbf{S}_{\delta(p)}$  is *Weyl-invariant* if it satisfies these conditions in  $\mathbf{S}_{\delta(p)}$ . We denote the torus  $\mathbb{R}/\tau\mathbb{Z}$  by  $\mathbb{T}$ , and usually identify it with  $[-\tau/2, \tau/2)$ . Set  $c_G = \frac{q \log q}{4\pi(q+1)}$ . It is well known [CMS3, formula (3), p. 55] that the inversion formula for the spherical Fourier transform may be written as follows:

$$(2.1) \quad f(x) = 2c_G q^{-|x|/2} \int_{\mathbb{T}} \tilde{f}(s) \mathbf{c}(-s)^{-1} q^{is|x|} ds.$$

Recall that the Fourier transformation  $\mathcal{F}$  on  $\mathbb{Z}$  has already been defined (see (1.1)). The corresponding inversion formula is

$$F(d) = \frac{1}{\tau} \int_{\mathbb{T}} \mathcal{F}F(s) q^{ids} ds \quad \forall d \in \mathbb{Z}.$$

Clearly  $\mathcal{F}F$  is  $\tau$ -periodic on  $\mathbb{R}$ . A distribution  $m$  on  $\mathbb{T}$  is said to be in  $\mathcal{M}_p(\mathbb{T})$  if convolution with  $\mathcal{F}^{-1}m$  defines a bounded operator on  $L^p(\mathbb{Z})$ . Note that  $\mathcal{M}_p(\mathbb{T})$  is contained in  $L^\infty(\mathbb{T})$ , because trivially  $\mathcal{M}_p(\mathbb{T})$  is contained in  $\mathcal{M}_2(\mathbb{T})$ , and  $\mathcal{M}_2(\mathbb{T})$  may be identified with  $L^\infty(\mathbb{T})$ .

A geodesic ray  $\gamma$  in  $\mathcal{T}$  is a one-sided sequence  $\{\gamma_n : n \in \mathbb{N}\}$  of points of  $\mathcal{T}$  such that  $d(\gamma_i, \gamma_j) = |i - j|$  for all nonnegative integers  $i$  and  $j$ . We say that  $x$  lies on  $\gamma$  if  $x = \gamma_n$  for some  $n$  in  $\mathbb{N}$ . Geodesic rays  $\{\gamma_n : n \in \mathbb{N}\}$  and  $\{\gamma'_n : n \in \mathbb{N}\}$  are identified if there exist integers  $i$  and  $j$  such that  $\gamma_n = \gamma'_{n+i}$  for all  $n$  greater than  $j$ ; this identification is an equivalence relation. We denote by  $\Omega$  the set of the equivalence classes, which we call boundary of  $\mathcal{T}$ , and by  $\Omega_x$  the set of all geodesic rays starting at  $x$ . Note that for every element  $\omega$  in  $\Omega$  there exists a unique representative geodesic ray in  $\Omega_x$ : we denote this geodesic ray by  $[x, \omega)$ . Given two geodesic rays  $\gamma^+ = [x, \omega^+)$  and  $\gamma^- = [x, \omega^-)$  with intersection  $\gamma^+ \cap \gamma^- = \{x\}$  we define the doubly infinite geodesic  $\gamma = \{\gamma_j : j \in \mathbb{Z}\}$  as follows:  $\gamma_j = \gamma_j^+$  if  $j \geq 0$  and  $\gamma_j = \gamma_j^-$  if  $j < 0$ . If  $\omega^+$  and  $\omega^-$  are two elements of  $\Omega$  there exists a unique (up to renumbering) geodesic  $\{\gamma_j : j \in \mathbb{Z}\}$  such that  $\omega^+$  and  $\omega^-$  are the equivalence classes of  $\{\gamma_j : j \in \mathbb{N}\}$  and  $\{\gamma_{-j} : j \in \mathbb{N}\}$  respectively. For brevity, we denote this geodesic by  $(\omega^+, \omega^-)$  disregarding the labels.

We fix a reference geodesic  $\gamma = (\omega^-, \omega^+)$  such that  $o$  lies on  $\gamma$ , and assume that  $\gamma$  is indexed so that  $\gamma_0 = o$ . Define the height function  $h$  (associated to  $\omega^+$ ) by the rule

$$h_{\omega^+}(x) := \lim_{i \rightarrow \infty} (i - d(x, \gamma_i)) \quad \forall x \in \mathcal{T}.$$

The level sets of the height function are called *horocycles* of  $\mathcal{T}$ .

We choose (once and for all) an isometry  $\sigma$  of  $\mathcal{T}$  that maps  $\gamma_i$  in  $\gamma_{i+1}$  for every  $i$ . Then, for  $j$  in  $\mathbb{Z}$ ,  $\sigma^j$  is an isometry of  $\mathcal{T}$  that maps  $\gamma_i$  to  $\gamma_{i+j}$ . The group  $G$  admits an Iwasawa-type decomposition  $G = NAG_o$ , investigated in [FTN, V]. Denote by  $A$  the subgroup of  $G$  generated by the one-step translation  $\sigma$  and by  $N$  the subgroup of  $G$  of all the elements that stabilise  $\omega^+$  and at least an element of  $\mathcal{T}$ . It is known that  $N$  can be characterised as the subgroup of  $G$  consisting in the elements that fix all the horocycles with respect to  $\omega^+$  [V, Lemma 3.1]. Furthermore, the orbit of an element  $x$  of  $\mathcal{T}$  under the action of  $N$  is the horocycle which contains  $x$  [V, Corollary 3.2].

We endow the totally disconnected group  $G$  with the Haar measure such that the mass of the open subgroup  $G_o$  is 1. Thus,

$$\int_G f'(g \cdot o) dg = \sum_{x \in \mathcal{T}} f(x)$$

for all finitely supported functions on  $\mathcal{T}$ . The reader can find much more on the group  $G$  in the book of A. Figà-Talamanca and C. Nebbia [FTN]. It is well known that the group  $N$  is unimodular; we normalise its Haar measure  $\mu$  by requiring that  $\mu(N \cap G_o) = 1$ , as in [V, Lemma 3.3]. The analogy between  $G$  and semisimple Lie groups of rank one is apparent in the following theorem [V, Theorem 3.5].

**Theorem 2.1.** *Let  $G$ ,  $N$ ,  $G_o$  and  $\sigma$  be as above. Then for every  $g$  in  $G$  there exist  $n$  in  $N$ ,  $j$  in  $\mathbb{Z}$  and  $g_o$  in  $G_o$  such that  $g = n\sigma^j g_o$ . Furthermore, if  $f$  is a continuous*

compactly supported function on  $G$ , then

$$\int_G f(g) dg = \int_N \sum_{j \in \mathbb{Z}} q^{-j} \int_{G_o} f(n\sigma^j g_o) dg_o d\mu(n).$$

We remark that, contrary to what happens in the case of noncompact symmetric spaces, there is a lack of uniqueness in this Iwasawa-type decomposition. Indeed, if  $g = n\sigma^j g_o = v\sigma^\ell h_o$ , then  $j = \ell$  and there exists  $n_o$  in  $N \cap G_o$  such that  $v = \sigma^j n_o \sigma^{-j}$  and  $h_o = n_o^{-1} g_o$  (see [V, Remark 3.6]).

Going back to the tree, a vertex  $x$  is of the form  $n\sigma^j \cdot o$ , with  $n$  in  $N$  and  $j$  in  $\mathbb{Z}$ . It is straightforward to prove that the height of  $x$  (with respect to  $\omega^+$ ) is simply  $j$ . The next lemma establishes a relation between the height of a point and its distance from the origin, and may be seen as an analogue of [I1, Lemma 3].

**Lemma 2.2.** *For every  $n$  in  $N$  and for every  $j$  in  $\mathbb{Z}$  such that  $j \leq d(n \cdot o, o)$*

$$d(n\sigma^j \cdot o, o) = d(n \cdot o, o) - j.$$

*In particular, this formula holds for every  $n$  in  $N$  and every nonpositive  $j$  in  $\mathbb{Z}$ .*

*Proof.* Write  $x$  instead of  $n\sigma^j \cdot o$ , and denote by  $\gamma_\ell$  the confluence point of  $[x, \omega^+)$  in  $\omega$ , i.e.  $[\gamma_\ell, \omega_+) = [x, \omega_+) \cap \omega$  (see also [CMS2, pag. 6]). Note that by definition  $\gamma_\ell$  lies on  $[x, \omega^+)$ , so  $\ell \geq j$ . We observe that such  $\gamma_\ell$  exists, because, by the definition of  $N$ , every element of this group fixes a geodesic ray equivalent to  $[\gamma_j, \omega_+)$ .

On a tree the union of two geodesic segments with one extreme in common (but no other point) is again a geodesic segment, so

$$(2.2) \quad d(x, o) = d(x, \gamma_\ell) + d(\gamma_\ell, o).$$

Note that  $d(\gamma_\ell, o)$  is the absolute value  $|\ell|$ , as  $o$  lies on the geodesic  $\gamma$ . Moreover,  $d(x, \gamma_\ell)$  is always equal to  $\ell - j$ , as we already noted that  $\ell \geq j$ .

Now we consider the cases where  $\ell \leq 0$  or  $\ell > 0$  separately. If  $\ell \leq 0$ , then  $n$  fixes the origin and (2.2) becomes

$$d(x, o) = (\ell - j) - \ell = -j = d(n \cdot o, o) - j.$$

Otherwise  $\ell > 0$ , and we have  $d(n \cdot o, o) = 2\ell$ , because  $n \cdot o$  belongs to the same horocycle as  $o$ . Hence (2.2) becomes

$$d(x, o) = (\ell - j) + \ell = 2\ell - j = d(n \cdot o, o) - j.$$

This concludes the proof of the lemma.  $\square$

Let  $N$  and  $A$  be the subgroups of  $G$  defined above, and consider the semi-direct product  $NA$ , where  $A$  acts on  $N$  by conjugation. By [V, Lemma 3.8] the modular function  $\Delta_{NA}$  of  $NA$  is given by

$$(2.3) \quad \Delta_{NA}(n\sigma^j) = q^{-j} \quad \forall n \in N \quad \forall j \in \mathbb{Z}.$$

By [V, Theorem 3.5], we may also identify the convolution between a  $G_o$ -right-invariant and  $G_o$ -bi-invariant functions on  $G$  with the convolution of the corresponding functions on the group  $NA$ . Explicitly, suppose that  $f$  is a  $G_o$ -right invariant function and that  $k$  is a  $G_o$ -bi-invariant function on  $G$ . Then, for  $v \in N$ ,  $j \in \mathbb{Z}$ , and  $g_o \in G_o$ , we have

$$\begin{aligned} f *_G k(v\sigma^j g_o) &= \int_N \sum_{\ell \in \mathbb{Z}} q^{-\ell} \int_{G_o} f(n\sigma^\ell h_o) k(h_o^{-1} \sigma^{-\ell} n^{-1} v\sigma^j g_o) dh_o d\mu(n) \\ &= \int_N \sum_{\ell \in \mathbb{Z}} \Delta_{NA}(n\sigma^\ell) f(n\sigma^\ell) k(\sigma^{-\ell} n^{-1} v\sigma^j) d\mu(n) \\ &= f *_NA k(v\sigma^j), \end{aligned}$$

where we have used the fact that  $G_o$  has total mass 1. By [V, Theorem 3.5], the norms of  $k$  in  $Cv_p(G)$  and in  $Cv_p(NA)$  coincide.

For  $p$  in  $[1, \infty)$ , we denote by  $Q_p : N \rightarrow \mathbb{R}$  the function defined by

$$(2.4) \quad Q_p(n) = q^{-|n \cdot o|/p}.$$

**Lemma 2.3.** *Suppose that  $p$  is in  $[1, 2)$ . Then the function  $n \mapsto |n \cdot o|^\ell Q_p(n)$  belongs to  $L^1(N)$  for each nonnegative integer  $\ell$ .*

*Proof.* For any nonnegative integer  $r$ , set  $T_r := \{v \in N : v \cdot o \in S_r(o)\}$ . By [V, Lemma 3.11],  $\mu(T_r)$  vanishes if  $r$  is odd, is equal to 1 if  $r = 0$ , and is equal to  $q^{r/2}$  if  $r$  is even and nonzero. Then, at least if  $\ell \geq 1$ ,

$$\begin{aligned} \int_N |n \cdot o|^\ell Q_p(n) d\mu(n) &= \sum_{r \geq 1} r^\ell q^{-r/p} \mu(T_r) + 1 \\ &= \sum_{j \geq 1} (2j)^\ell q^{j(1-2/p)} + 1, \end{aligned}$$

which is convergent, because  $1 \leq p < 2$ , as required.  $\square$

### 3. A GENERAL TRANSFERENCE RESULT

Denote by  $\Gamma$  a locally compact group, with left Haar measure  $\lambda$ . Integration will be with respect to  $\lambda$ , unless otherwise specified. We denote by  $\Delta_\Gamma$  the modular function on  $\Gamma$ , i.e. the Radon–Nykodim derivative of  $\lambda$  with respect to the right Haar measure. Given “nice” functions  $f$  and  $\kappa$  on  $\Gamma$ , their convolution on  $\Gamma$  is defined by

$$f * \kappa(x) = \int_\Gamma f(y) \kappa(y^{-1}x) d\lambda(y),$$

Recall the following basic convolution inequality

$$\|f * \kappa\|_{L^p(\Gamma)} \leq \|f\|_{L^p(\Gamma)} \|\Delta_\Gamma^{-1/p'} \kappa\|_{L^1(\Gamma)}.$$

(see e.g. [HR, Corollary 20.14 (ii) and (iv)]). We denote by  $Cv_p(\Gamma)$  the space of bounded *right* convolutors of  $L^p(\Gamma)$ . This space is equipped with the operator norm

$$\|\kappa\|_{Cv_p(\Gamma)} = \sup_{\|f\|_{L^p(\Gamma)}=1} \|f * \kappa\|_{L^p(\Gamma)}.$$

In this section we assume that the locally compact group  $\Gamma$  is the semi-direct product of two groups  $M$  and  $H$ , where  $M$  is normal in  $\Gamma$  and  $H$  acts on  $M$  by conjugation. Right Haar measures on  $M$  and  $H$  will be denoted by  $dn$  and  $dh$ , respectively. Then  $dg = dn dh$  is a right Haar measure on  $\Gamma$ . We denote by  $\Delta_M$  and  $\Delta_H$  the modular functions of  $M$  and  $H$ , so that  $d\lambda(n) = \Delta_M(n) dn$  and  $d\lambda(h) = \Delta_H(h) dh$  are left Haar measures on  $M$  and  $H$ , respectively. Note that there is a slight abuse of notation here, for  $\lambda$  denotes both a left invariant measure on  $H$  and a left Haar measure on  $M$ .

For  $h$  in  $H$  and  $n$  in  $M$ , denote by  $n^h$  the conjugate  $hnh^{-1}$ , and by  $\mathcal{D}(h)^{-1}$  the Radon–Nykodim derivative  $d(n^h)/dn$ . It is not hard to check that  $\mathcal{D}$  is an homomorphism of  $H$ , i.e.  $\mathcal{D}(hh') = \mathcal{D}(h)\mathcal{D}(h')$  for every  $h$  and  $h'$  in  $H$ .

*Remark 3.1.* Observe that  $\mathcal{D}(h)^{-1} = d\lambda(n^h)/d\lambda(n)$ . Indeed, note that the conjugation by  $h$  commutes with the inversion on  $M$ , i.e.  $(n^h)^{-1} = (n^{-1})^h$ . Hence

$$\int_M f(n^h) d\lambda(n) = \int_M f((n^{-1})^h)^{-1} d\lambda(n).$$

Now, the inversion in  $M$  transforms the left Haar measure to the right Haar measure, and conversely. Thus,

$$\begin{aligned} \int_M f((n^{-1})^h)^{-1} d\lambda(n) &= \int_M f((v^h)^{-1}) dv \\ &= \mathcal{D}(h) \int_M f(n^{-1}) dn \\ &= \mathcal{D}(h) \int_M f(v) d\lambda(v). \end{aligned}$$

This fact will be used repeatedly in the proof of Theorem 3.3.

*Remark 3.2.* Observe that  $\mathcal{D}$  may be extended to a homomorphism on the whole group  $\Gamma$ , by setting  $\mathcal{D}(nh) := \mathcal{D}(h)$  for all  $n$  in  $M$  and  $h$  in  $H$ . Recall that  $(nh)(n_1h_1) = nn_1^h hh_1$ . Thus,

$$\begin{aligned} \mathcal{D}((nh)(n_1h_1)) &= \mathcal{D}(nn_1^h hh_1) = \mathcal{D}(hh_1) = \mathcal{D}(h)\mathcal{D}(h_1) \\ &= \mathcal{D}(nh)\mathcal{D}(n_1h_1). \end{aligned}$$

This observation applies to any homomorphism of  $H$ .

It is well known that  $\mathcal{D}(h)\Delta_M(n)\Delta_H(h)dn dh$  is a left Haar measure on  $\Gamma$  (see [HR, p. 211]), and that the following integral formulae hold

$$\begin{aligned} \int_\Gamma f(g) d\lambda(g) &= \int_M \int_H f(nh) \mathcal{D}(h) \Delta_M(n) \Delta_H(h) dn dh \\ &= \int_H \int_M f(hn) dn dh. \end{aligned}$$

The space  $L^1(M; C v_p(H))$  is the set of all distributions  $\kappa$  on  $\Gamma$  such that for (almost) every  $n$  in  $M$  the distribution  $\kappa(n\cdot)$  induces a bounded convolution operator on  $L^p(H)$ ,

and the function  $n \mapsto \|\kappa(n \cdot)\|_{Cv_p(H)}$  is in  $L^1(M)$ . The space  $L^1(M; Cv_p(H))$  is endowed with the norm

$$(3.1) \quad \|\kappa\|_{L^1(M; Cv_p(H))} := \int_M \|\kappa(n \cdot)\|_{Cv_p(H)} d\lambda(n).$$

**Theorem 3.3.** *Suppose that  $p$  is in  $(1, \infty)$  and that  $\Delta_M^{-1/p'} \kappa$  belongs to  $L^1(M; Cv_p(H))$ . Then the operator  $f \mapsto f * (\mathcal{D}^{-1/p} \kappa)$  is bounded on  $L^p(\Gamma)$ , and*

$$\|f * (\mathcal{D}^{-1/p} \kappa)\|_{L^p(\Gamma)} \leq \|f\|_{L^p(\Gamma)} \|\Delta_M^{-1/p'} \kappa\|_{L^1(M; Cv_p(H))}.$$

Notice that

$$\|\Delta_M^{-1/p'} \kappa\|_{L^1(M; Cv_p(H))} = \int_M \|\kappa(n \cdot)\|_{Cv_p(H)} \Delta_M(n)^{-1/p'} d\lambda(n).$$

*Proof.* Notice that  $(nh)^{-1} n_1 h_1 = h^{-1} n^{-1} n_1 h_1 = (n^{-1} n_1)^{h^{-1}} h^{-1} h_1$ . Thus,

$$f * (\mathcal{D}^{-1/p} \kappa)(n_1 h_1) = \int_M \int_H f(nh) \mathcal{D}^{-1/p}(h^{-1} h) \kappa((n^{-1} n_1)^{h^{-1}} h^{-1} h_1) \mathcal{D}(h) d\lambda(n) d\lambda(h).$$

We change variables  $((n^{-1} n_1)^{h^{-1}} = m^{-1})$  in the integral over  $M$ . Then  $m^{-1} = h^{-1} n^{-1} n_1 h$ , so that  $m = h^{-1} n_1^{-1} n h = (n_1^{-1} n)^{h^{-1}}$ , and

$$\frac{d\lambda(m)}{d\lambda(n)} = \frac{d\lambda((n_1^{-1} n)^{h^{-1}})}{d\lambda(n_1^{-1} n)} \frac{d\lambda(n_1^{-1} n)}{d\lambda(n)} = \mathcal{D}(h).$$

The last equality follows from the fact that  $\mathcal{D}$  is a homomorphism (whence  $\mathcal{D}(h^{-1})^{-1} = \mathcal{D}(h)$ ), and from the left invariance of  $\lambda$ . We conclude that  $d\lambda(n) = \mathcal{D}^{-1}(h) d\lambda(m)$ , whence

$$f * (\mathcal{D}^{-1/p} \kappa)(n_1 h_1) = \int_M d\lambda(m) \int_H f(n_1 m^h h) \mathcal{D}^{-1/p}(h^{-1} h_1) \kappa(m^{-1} h^{-1} h_1) d\lambda(h).$$

We set  $U(n_1, m, h) := f(n_1 m^h h)$ , and view the inner integral as the convolution on  $H$  between  $U(n_1, m, \cdot)$  and  $\mathcal{D}^{-1/p}(\cdot) \kappa(m^{-1} \cdot)$ , evaluated at the point  $h_1$ . Therefore

$$\begin{aligned} & \|f * (\mathcal{D}^{-1/p} \kappa)\|_{L^p(\Gamma)} \\ &= \left( \int_M \int_H |f * (\mathcal{D}^{-1/p} \kappa)(n_1 h_1)|^p \mathcal{D}(h_1) d\lambda(h_1) d\lambda(n_1) \right)^{1/p} \\ &= \left\| \left\| \int_M [U(n_1, m, \cdot) *_H (\mathcal{D}^{-1/p} \kappa)(m^{-1} \cdot)](h_1) \mathcal{D}^{1/p}(h_1) d\lambda(m) \right\|_{L^p(H)} \right\|_{L^p(M)} \end{aligned}$$

where the  $L^p(M)$  norm is taken with respect to the left Haar measure of  $M$  and the variable  $n_1$ . Observe that the argument of the integral over  $M$  above may be written as

$$\int_H U(n_1, m, h) \mathcal{D}^{-1/p}(h^{-1} h_1) \kappa(m^{-1} h^{-1} h_1) \mathcal{D}^{1/p}(h_1) dh.$$

Since  $\mathcal{D}$  is an homomorphism, this simplifies to

$$\int_H U(n_1, m, h) \mathcal{D}^{1/p}(h) \kappa(m^{-1} h^{-1} h_1) dh = [(\mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot)) *_H \kappa(m^{-1} \cdot)](h_1).$$



Therefore, by Minkowski's integral inequality,

$$(3.2) \quad \|f * (\mathcal{D}^{-1/p} \kappa)\|_{L^p(\Gamma)} \leq \int_M d\lambda(m) \left\| \left\| (\mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot)) *_H \kappa(m^{-1} \cdot) \right\|_{L^p(H)} \right\|_{L^p(M)}.$$

By assumption, for every  $m$  in  $M$  the function  $\kappa(m^{-1} \cdot)$  is in  $Cv_p(H)$ , so that

$$\begin{aligned} & \left\| (\mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot)) *_H \kappa(m^{-1} \cdot) \right\|_{L^p(H)} \\ & \leq \left\| \mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot) \right\|_{L^p(H)} \left\| \kappa(m^{-1} \cdot) \right\|_{Cv_p(H)}, \end{aligned}$$

and

$$\begin{aligned} & \left\| \left\| (\mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot)) *_H \kappa(m^{-1} \cdot) \right\|_{L^p(H)} \right\|_{L^p(M)} \\ & \leq \left\| \left\| \mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot) \right\|_{L^p(H)} \right\|_{L^p(M)} \left\| \kappa(m^{-1} \cdot) \right\|_{Cv_p(H)}. \end{aligned}$$

Observe that

$$\left\| \left\| \mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot) \right\|_{L^p(H)} \right\|_{L^p(M)} = \left[ \int_M \int_H |f(n_1 m^h h)|^p \mathcal{D}(h) d\lambda(n_1) d\lambda(h) \right]^{1/p}.$$

We change variables ( $n_1 m^h = n$ ) in the inner integral, write  $n_1 m^h = (n_1^{h^{-1}} m)^h$ , and observe that

$$\begin{aligned} \frac{d\lambda(n)}{d\lambda(n_1)} &= \frac{d\lambda((n_1^{h^{-1}} m)^h)}{d\lambda(n_1^{h^{-1}} m)} \frac{d\lambda(n_1^{h^{-1}} m)}{d\lambda(n_1^{h^{-1}})} \frac{d\lambda(n_1^{h^{-1}})}{d\lambda(n_1)} \\ &= \mathcal{D}(h)^{-1} \Delta_M(m) \mathcal{D}(h^{-1})^{-1} \\ &= \Delta_M(m), \end{aligned}$$

i.e.,  $d\lambda(n_1) = \Delta_M^{-1}(m) d\lambda(n)$ . Then

$$\left\| \left\| \mathcal{D}^{1/p}(\cdot) U(n_1, m, \cdot) \right\|_{L^p(H)} \right\|_{L^p(M)} = \Delta_M^{-1/p}(m) \|f\|_{L^p(\Gamma)}.$$

By combining this and (3.2), we obtain that

$$(3.3) \quad \begin{aligned} \|f * \kappa\|_{L^p(\Gamma)} &\leq \|f\|_{L^p(\Gamma)} \int_M \left\| \kappa(m^{-1} \cdot) \right\|_{Cv_p(H)} \Delta_M^{-1/p}(m) d\lambda(m) \\ &= \|f\|_{L^p(\Gamma)} \int_M \left\| \kappa(m \cdot) \right\|_{Cv_p(H)} \Delta_M^{1/p}(m) dm; \end{aligned}$$

the equality above is a consequence of the change of variables ( $m^{-1} \mapsto m$ ), which transforms the left Haar measure into the right Haar measure. Finally,

$$\Delta_M^{1/p}(m) dm = \Delta_M^{-1/p'}(m) \Delta_M(m) dm = \Delta_M^{-1/p'}(m) d\lambda(m),$$

which, together with (3.3), gives

$$\|f * \kappa\|_{L^p(\Gamma)} \leq \|f\|_{L^p(\Gamma)} \left\| \Delta_M^{-1/p'} \kappa \right\|_{L^1(M; Cv_p(H))},$$

as required.  $\square$

We shall apply Theorem 3.3 when  $M$  is unimodular. For the reader's convenience, we state the corresponding results in the following corollary.

**Corollary 3.4.** *Suppose that  $p$  is in  $(1, \infty)$  and that  $M$  is unimodular. Assume that  $\kappa$  belongs to  $L^1(M; Cv_p(H))$ . Then the operator  $f \mapsto f * (\mathcal{D}^{-1/p} \kappa)$  is bounded on  $L^p(\Gamma)$ . Furthermore,*

$$\|f * (\mathcal{D}^{-1/p} \kappa)\|_{L^p(\Gamma)} \leq \|f\|_{L^p(\Gamma)} \|\kappa\|_{L^1(M; Cv_p(H))}.$$

#### 4. $L^p$ SPHERICAL MULTIPLIERS ON A TREE

In this section we prove our main result. We first need a lemma on convolutors of  $L^p(\mathbb{Z})$  whose Fourier transforms extend to holomorphic functions in a strip. For each positive  $\varepsilon$ , we set  $\Sigma_\varepsilon := \{z \in \mathbb{C} : -\varepsilon < \operatorname{Im} z < 0\}$ .

**Theorem 4.1.** *Suppose that  $p$  is in  $[1, \infty)$ , that  $\varphi$  is in  $Cv_p(\mathbb{Z})$ , and that  $\mathcal{F}\varphi$  extends to a bounded holomorphic function in the strip  $\Sigma_\varepsilon$  for some positive  $\varepsilon$ . Then*

$$\|\varphi \mathbf{1}_{[J, \infty)}\|_{Cv_p(\mathbb{Z})} \leq \|\varphi\|_{Cv_p(\mathbb{Z})} + \left(\frac{1}{q^\varepsilon - 1} + J\right) \|\mathcal{F}\varphi\|_{H^\infty(\Sigma_\varepsilon)} \quad \forall J \in \mathbb{N}.$$

*Remark 4.2.* The conclusion fails for a generic convolutor of  $L^p(\mathbb{Z})$ . For instance, it is well known that the function  $\varphi(j) := j^{-1} \mathbf{1}_{\mathbb{Z} \setminus \{0\}}(j)$  is in  $Cv_p(\mathbb{Z})$  for all  $p$  in  $(1, \infty)$ . However,  $\varphi \mathbf{1}_{[0, \infty)}$  is not in  $Cv_p(\mathbb{Z})$  for any  $p$  in  $(1, \infty)$ . Indeed, if  $\varphi \mathbf{1}_{[0, \infty)}$  were a convolutor of  $L^p(\mathbb{Z})$ , then it would be a finite measure, because  $\varphi \mathbf{1}_{[0, \infty)}$  is nonnegative and  $\mathbb{Z}$  is amenable. This contradicts the fact that  $\varphi \mathbf{1}_{[0, \infty)}$  is nonintegrable on  $\mathbb{Z}$ .

*Proof.* Observe that  $\mathcal{F}\varphi$  is  $\tau$ -periodic in the strip  $\Sigma_\varepsilon$ . A standard argument based on Cauchy's theorem allows us to move the path of integration from  $[-\tau/2, \tau/2]$  to  $[-\tau/2, \tau/2] - i\varepsilon$  (the integrals over the vertical sides of the rectangle  $[-\tau/2, \tau/2] \times [-\varepsilon, 0]$  cancel out by periodicity), and obtain that

$$\varphi(j) = \frac{1}{\tau} \int_{\mathbb{T}} \mathcal{F}\varphi(s) q^{ijs} ds = \frac{1}{\tau} \int_{\mathbb{T}} \mathcal{F}\varphi(s - i\varepsilon) q^{ij(s - i\varepsilon)} ds.$$

Hence

$$|\varphi(j)| \leq \|\mathcal{F}\varphi\|_{H^\infty(\Sigma_\varepsilon)} q^{j\varepsilon} \quad \forall j \in \mathbb{Z},$$

so that  $\varphi \mathbf{1}_{(-\infty, -1]}$  is integrable on  $\mathbb{Z}$ , and

$$\|\varphi \mathbf{1}_{(-\infty, -1]}\|_{Cv_p(\mathbb{Z})} \leq \|\varphi \mathbf{1}_{(-\infty, -1]}\|_{L^1(\mathbb{Z})} \leq \frac{1}{q^\varepsilon - 1} \|\mathcal{F}\varphi\|_{H^\infty(\Sigma_\varepsilon)}.$$

Furthermore, trivially

$$|\varphi(j)| \leq \|\mathcal{F}\varphi\|_{L^\infty(\mathbb{T})} \quad \forall j \in \mathbb{Z},$$

whence the function  $\varphi \mathbf{1}_{[0, J-1]}$  satisfies the estimate

$$\|\varphi \mathbf{1}_{[0, J-1]}\|_{Cv_p(\mathbb{Z})} \leq \|\varphi \mathbf{1}_{[0, J-1]}\|_{L^1(\mathbb{Z})} \leq J \|\mathcal{F}\varphi\|_{L^\infty(\mathbb{T})}.$$

As a consequence

$$\begin{aligned} \|\varphi \mathbf{1}_{[J,\infty)}\|_{Cv_p(\mathbb{Z})} &\leq \|\varphi\|_{Cv_p(\mathbb{Z})} + \|\varphi \mathbf{1}_{(-\infty,-1]}\|_{Cv_p(\mathbb{Z})} + \|\varphi \mathbf{1}_{[0,J-1]}\|_{L^1(\mathbb{Z})} \\ &\leq \|\varphi\|_{Cv_p(\mathbb{Z})} + \frac{1}{q^\varepsilon - 1} \|\mathcal{F}\varphi\|_{H^\infty(\Sigma_\varepsilon)} + J \|\mathcal{F}\varphi\|_{L^\infty(\mathbb{T})} \\ &\leq \|\varphi\|_{Cv_p(\mathbb{Z})} + \left(\frac{1}{q^\varepsilon - 1} + J\right) \|\mathcal{F}\varphi\|_{H^\infty(\Sigma_\varepsilon)}, \end{aligned}$$

as required.  $\square$

Recall that  $\delta(p) = |1/p - 1/2|$  and that  $\mathbf{S}_{\delta(p)} = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \delta(p)\}$ . The main result of this paper is the following.

**Theorem 4.3.** *Suppose that  $p$  is in  $[1, \infty) \setminus \{2\}$ , and that  $k$  is a radial function on  $\mathcal{T}$ . The following are equivalent:*

- (i)  $k$  is in  $Cv_p(\mathcal{T})$ ;
- (ii)  $\tilde{k}$  is a holomorphic Weyl-invariant function on  $\mathbf{S}_{\delta(p)}$ , and  $\tilde{k}_{\delta(p)}$  is in  $\mathcal{M}_p(\mathbb{T})$ .

Furthermore, there exists positive constants  $c$  and  $C$ , independent of  $k$ , such that

$$c \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})} \leq \|k\|_{Cv_p(\mathcal{T})} \leq C \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})}.$$

*Proof.* It is known that (i) implies (ii) and that the left hand inequality above holds (see [CMS3, Theorem 2.1]).

Thus, it remains to show that (ii) implies (i). Observe that it suffices to prove the result in the case where  $p$  is in  $[1, 2)$ . Indeed, if  $p$  is in  $(2, \infty)$ , and  $\tilde{k}_{\delta(p)}$  is in  $\mathcal{M}_p(\mathbb{T})$ , then  $\tilde{k}_{\delta(p)}$  is also in  $\mathcal{M}_{p'}(\mathbb{T})$ . Since  $p'$  is in  $(1, 2)$ ,  $k$  is in  $Cv_{p'}(\mathcal{T})$ . A straightforward duality argument then shows that  $k$  is in  $Cv_p(\mathcal{T})$ , as required. Here we use the fact that  $k$  is radial.

Henceforth we assume that  $p$  is in  $[1, 2)$ . By the inversion formula (2.1),

$$(4.1) \quad k(x) = 2c_G q^{-|x|/2} \int_{\mathbb{T}} \tilde{k}(s) \mathbf{c}(-s)^{-1} q^{is|x|} ds \quad \forall x \in \mathcal{T}.$$

The integrand in (4.1) above is  $\tau$ -periodic, and holomorphic in the rectangle  $(-\tau/2, \tau/2) \times (-\delta(p), \delta(p))$ . A standard argument based on Cauchy's theorem allows us to move the path of integration from  $[-\tau/2, \tau/2]$  to  $[-\tau/2, \tau/2] + i\delta(p)$  (the integrals over the vertical sides of the rectangle  $[-\tau/2, \tau/2] \times [0, \delta(p)]$  cancel out by periodicity), and obtain that

$$k(x) = 2c_G q^{-|x|/p} \int_{\mathbb{T}} \tilde{k}(s + i\delta(p)) \mathbf{c}(-s - i\delta(p))^{-1} q^{is|x|} ds.$$

We write  $(\tilde{k}\check{\mathbf{c}}^{-1})_{\delta(p)}$  instead of  $\tilde{k}(\cdot + i\delta(p)) \mathbf{c}(\cdot - i\delta(p))^{-1}$ , and introduce the function  $\varphi$  on  $\mathbb{Z}$ , defined by

$$\varphi(\ell) = 2c_G \int_{\mathbb{T}} (\tilde{k}\check{\mathbf{c}}^{-1})_{\delta(p)}(s) q^{is\ell} ds.$$

Then

$$(4.2) \quad k(x) = q^{-|x|/p} \varphi(|x|).$$

Suppose first that  $p = 1$ . We must prove that  $k$  belongs to  $L^1(\mathcal{T})$ . Since

$$\|k\|_{L^1(\mathcal{T})} = \sum_{x \in \mathcal{T}} q^{-|x|} |\varphi(|x|)| = |\varphi(0)| + \frac{q+1}{q} \sum_{d=1}^{\infty} |\varphi(d)|,$$

it suffices to prove that  $\varphi$  is in  $L^1(\mathbb{Z})$ . Obviously,

$$\varphi = 2c_G \tau \mathcal{F}^{-1}(\tilde{k} \check{\mathbf{c}}^{-1})_{\delta(p)} = 2c_G \tau \mathcal{F}^{-1} \tilde{k}_{\delta(p)} *_z \mathcal{F}^{-1}(\check{\mathbf{c}}^{-1})_{\delta(p)}.$$

Since  $(\check{\mathbf{c}}^{-1})_{\delta(p)}$  is smooth on  $\mathbb{T}$ , its inverse Fourier transform  $\mathcal{F}^{-1}(\check{\mathbf{c}}^{-1})_{\delta(p)}$  is in  $L^1(\mathbb{Z})$ , by classical Fourier analysis. Furthermore  $\mathcal{F}^{-1} \tilde{k}_{\delta(p)}$  is in  $L^1(\mathbb{Z})$  by assumption. Therefore  $\varphi$  is in  $L^1(\mathbb{Z})$  and the proof in the case where  $p = 1$  is complete.

Now assume that  $p$  is in  $(1, 2)$ . Denote by  $\chi^+$  and  $\chi^-$  the functions on  $\mathcal{T}$  defined by

$$\chi^+(v\sigma^j \cdot o) = \mathbf{1}_{[0, \infty)}(j) \quad \text{and} \quad \chi^-(v\sigma^j \cdot o) = \mathbf{1}_{(-\infty, -1]}(j),$$

where  $v \in N$  and  $j \in \mathbb{Z}$ . Clearly

$$k = k\chi^- + k\chi^+.$$

It is convenient to view the kernel  $k$  as a function on the group  $NA$ . In particular, we use formula (4.2), the definition of  $\chi^-$  above, change variables (see Lemma 2.2), recall that  $Q_p(v) = q^{-|v \cdot o|/p}$  (see formula (2.4)), and obtain that

$$\begin{aligned} (k\chi^-)(v\sigma^j \cdot o) &= q^{-(|v \cdot o| - j)/p} \varphi(|v \cdot o| - j) \mathbf{1}_{(-\infty, -1]}(j) \\ (4.3) \quad &= q^{j/p} Q_p(v) \varphi(|v \cdot o| - j) \mathbf{1}_{(-\infty, -1]}(j). \end{aligned}$$

*Step I: analysis of  $k\chi^-$ .* Observe that

$$\begin{aligned} |\varphi(j)| &\leq 2c_G \int_{\mathbb{T}} |(\tilde{k} \check{\mathbf{c}}^{-1})_{\delta(p)}(s)| \, ds \\ &\leq 2c_G \tau \|\tilde{k} \check{\mathbf{c}}^{-1}\|_{\delta(p)}_{\infty} \\ &\leq 2c_G \tau \|\tilde{k} \check{\mathbf{c}}^{-1}\|_{L^\infty(\mathbf{S}_{\delta(p)})} \end{aligned}$$

for every integer  $j$ . As a consequence we obtain the following pointwise bound

$$(4.4) \quad |(k\chi^-)(v\sigma^j \cdot o)| \leq 2c_G \tau \|\tilde{k} \check{\mathbf{c}}^{-1}\|_{L^\infty(\mathbf{S}_{\delta(p)})} q^{j/p} \mathbf{1}_{(-\infty, -1]}(j) Q_p(v),$$

which we record for later use. Formula (4.3) and Corollary 3.4 (with  $\Gamma = NA$ ,  $\mathcal{D}(v\sigma^j) = q^{-j}$  and  $\kappa = \mathcal{D}^{1/p} k\chi^-$ ) imply that

$$\begin{aligned} \|k\chi^-\|_{Cv_p(NA)} &\leq \|\mathcal{D}^{1/p} k\chi^-\|_{L^1(N; Cv_p(\mathbb{Z}))} \\ &= \int_N \|\varphi(|v \cdot o| - \cdot) \mathbf{1}_{(-\infty, -1]}\|_{Cv_p(\mathbb{Z})} Q_p(v) \, dv. \end{aligned}$$

Clearly the norm in  $Cv_p(\mathbb{Z})$  is translation invariant; it is then straightforward to check that

$$\|\varphi(|v \cdot o| - \cdot) \mathbf{1}_{(-\infty, -1]}\|_{Cv_p(\mathbb{Z})} = \|\varphi \mathbf{1}_{[|v \cdot o|, \infty)}\|_{Cv_p(\mathbb{Z})}.$$

By Theorem 4.1 (with  $2\delta(p)$  in place of  $\varepsilon$ , and  $|v \cdot o|$  in place of  $J$ )

$$\|\varphi \mathbf{1}_{[|v \cdot o|, \infty)}\|_{Cv_p(\mathbb{Z})} \leq \|\varphi\|_{Cv_p(\mathbb{Z})} + \left( \frac{1}{q^{2\delta(p)} - 1} + |v \cdot o| \right) \|\mathcal{F}\varphi\|_{H^\infty(\Sigma_{2\delta(p)})}.$$

By definition of  $\varphi$  and of the multiplier norm,  $\|\varphi\|_{Cv_p(\mathbb{Z})} = 2c_G \tau \|\tilde{k}\check{\mathbf{c}}^{-1}\|_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})}$ . Notice that the function  $\check{\mathbf{c}}_{\delta(p)}^{-1}$  is smooth on  $\mathbb{R}$ , and never vanishes. Therefore there exists a constant  $C$  such that

$$\|(\tilde{k}\check{\mathbf{c}}^{-1})_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})} \leq C \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})}.$$

Furthermore,

$$\begin{aligned} (2c_G \tau)^{-1} \|\mathcal{F}\varphi\|_{H^\infty(\Sigma_{2\delta(p)})} &= \|(\tilde{k}\check{\mathbf{c}}^{-1})_{\delta(p)}\|_{H^\infty(\Sigma_{2\delta(p)})} \\ &= \max \left[ \|(\tilde{k}\check{\mathbf{c}}^{-1})_{\delta(p)}\|_{L^\infty(\mathbb{T})}, \|(\tilde{k}\check{\mathbf{c}}^{-1})_{-\delta(p)}\|_{L^\infty(\mathbb{T})} \right] \\ &\leq \|\check{\mathbf{c}}^{-1}\|_{H^\infty(\mathbf{s}_{\delta(p)})} \max \left[ \|\tilde{k}_{\delta(p)}\|_{L^\infty(\mathbb{T})}, \|\tilde{k}_{-\delta(p)}\|_{L^\infty(\mathbb{T})} \right] \\ &= \|\check{\mathbf{c}}^{-1}\|_{H^\infty(\mathbf{s}_{\delta(p)})} \|\tilde{k}_{\delta(p)}\|_{L^\infty(\mathbb{T})} \\ &\leq \|\check{\mathbf{c}}^{-1}\|_{H^\infty(\mathbf{s}_{\delta(p)})} \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})}; \end{aligned}$$

we have used the Weyl-invariance of  $\tilde{k}$  in the last equality above. By combining the formulae above, we obtain that

$$\|k\chi^-\|_{Cv_p(NA)} \leq C \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})} \int_N (1 + |v \cdot o|) Q_p(v) dv \leq C \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})};$$

the last inequality follows from Lemma 2.3.

*Step II: analysis of  $k\chi^+$ .* Recall that the modular function on  $NA$  is  $\Delta_{NA}(v\sigma^j) = q^{-j}$  (see (2.3)). Thus,

$$\begin{aligned} \|\Delta_{NA}^{-1/p'} k\chi^+\|_{L^1(NA)} &= \sum_{j \in \mathbb{Z}} q^{-j} q^{j/p'} \int_N |k\chi^+(v\sigma^j \cdot o)| d\mu(v) \\ &= \sum_{j \geq 0} q^{-j/p} \int_N |k(v\sigma^j \cdot o)| d\mu(v). \end{aligned}$$

Recall that the Abel transform (see [CMS2]) of  $|k|$  is defined by

$$\mathcal{A}(|k|)(j) = q^{-j/2} \int_N |k(v\sigma^j \cdot o)| d\mu(v).$$

By [CMS2, Theorem 2.5], the Abel transform of  $|k|$  is an even function on  $\mathbb{Z}$ , equivalently

$$\int_N |k(v\sigma^j \cdot o)| d\mu(v) = q^j \int_N |k(v\sigma^{-j} \cdot o)| d\mu(v).$$

Altogether, we see that

$$\|\Delta_{NA}^{-1/p'} k\chi^+\|_{L^1(NA)} = \sum_{j \geq 0} q^{j/p'} \int_N |k(v\sigma^{-j} \cdot o)| d\mu(v).$$

By the pointwise bound (4.4) the right hand side is dominated by

$$2c_G \tau \|\tilde{k}\check{\mathbf{c}}^{-1}\|_{L^\infty(\mathbf{S}_{\delta(p)})} \sum_{j \geq 0} q^{j/p'} q^{-j/p} \int_N Q_p(v) d\mu(v).$$

Now, the integral over  $N$  is convergent, because  $p > 1$  (see Lemma 2.3), and so is the series, because  $p < 2 < p'$ . Therefore

$$(4.5) \quad \|\Delta_{NA}^{-1/p'} k \chi^+\|_{L^1(NA)} \leq C \|\tilde{k}\check{\mathbf{c}}^{-1}\|_{H^\infty(\mathbf{S}_{\delta(p)})} \leq C \|\tilde{k}\|_{H^\infty(\mathbf{S}_{\delta(p)})} :$$

the last inequality follows from the fact that  $\check{\mathbf{c}}^{-1}$  is bounded on  $\mathbf{S}_{\delta(p)}$ . Since  $\tilde{k}$  is bounded and Weyl-invariant on  $\mathbf{S}_{\delta(p)}$ ,

$$\|\tilde{k}\|_{H^\infty(\mathbf{S}_{\delta(p)})} \leq \|\tilde{k}_{\delta(p)}\|_{L^\infty(\mathbb{T})} \leq \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})}.$$

*Step III: conclusion.* By combining the estimates proved in *Step I* and *Step II*, we see that there exists a constant  $C$ , independent of  $k$ , such that

$$\begin{aligned} \|k\|_{Cv_p(NA)} &\leq \|k\chi^+\|_{Cv_p(NA)} + \|k\chi^-\|_{Cv_p(NA)} \\ &\leq C \|\tilde{k}_{\delta(p)}\|_{\mathcal{M}_p(\mathbb{T})}. \end{aligned}$$

Since  $k$  is radial on  $\mathcal{T}$ ,  $\|k\|_{Cv_p(NA)} = \|k\|_{Cv_p(\mathcal{T})}$ . Thus,  $k$  is in  $Cv_p(\mathcal{T})$  and the required norm estimate holds.

This concludes the proof of the theorem.  $\square$

## REFERENCES

- [CSt] J.-L. Clerc and E. M. Stein,  $L^p$  multipliers for noncompact symmetric spaces, *Proc. Nat. Acad. Sci. U. S. A.* **71** (1974), 3911–3912.
- [CMS1] M. Cowling, S. Meda and A.G. Setti, Estimates for Functions of the Laplace Operator on Homogeneous Trees, *Trans. Am. Math. Soc.* **352**, No. 9, (2000), 4271–4293.
- [CMS2] M. Cowling, S. Meda, A. G. Setti, An overview of harmonic analysis on the group of isometries of a homogeneous tree, *Exposition. Math.* **16** (1998), 385–423.
- [CMS3] M. Cowling, S. Meda, A. G. Setti, Invariant operators on function spaces on homogeneous trees, *Colloq. Math.*, **80** (1999), 53–61.
- [CS] M. Cowling and A.G. Setti, The range of the Helgason-Fourier transformation on homogeneous trees, *Bull. Austral. Math. Soc.* **59**, (1999), 237–246.
- [FTN] A. Figà Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, London Math. Society Lecture Notes Series, n. **162**, Cambridge University Press, 1991.
- [FTP] A. Figà Talamanca and M. Picardello, *Harmonic Analysis on Free Groups*, Lecture Notes in Pure and Applied Mathematics, n. **87**, Marcel Dekker, 1983.
- [HR] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis* vol. 1, A Series of Comprehensive Studies in Mathematics n. **115**, Springer-Verlag, 1979.
- [I1] A.D. Ionescu, Singular integrals on symmetric spaces of real rank one, *Duke Math. J.* **114** (2002), 101–122.
- [MS1] G. Medolla and A.G. Setti, The wave equation on homogeneous trees, *Ann. Mat. Pura Appl.* **176** (1999), 1–27.
- [MS2] G. Medolla and A.G. Setti, Long time heat diffusion on homogeneous trees, *Proc. Amer. Math. Soc.* **128** (1999), 1733–1742.

- [N] C. Nebbia, Groups of isometries of a tree and the Kunze-Stein phenomenon, *Pacific J. Math.* **133** (1988), 141–149.
- [P] T. Pytlik, Radial convolutors on free groups, *Studia Math.* **78** (1984), 178–183.
- [Sz] R. Szwarc, Convolution operators of weak type  $(p, p)$ , which are not of strong type  $(p, p)$ , *Proc. Amer. Math. Soc.*, **89**, No. 1. (1983), 184–185.
- [Se1] A.G. Setti,  $L^p$ - $L^r$  estimates for the Poisson semigroup on homogeneous trees, *J. Austral. Math. Soc. (Series A)*, **64** (1998), 20–32.
- [Se2] A.G. Setti,  $L^p$  and operator norm estimates for the complex time heat semigroup on homogeneous trees, *Trans. Amer. Math. Soc.*, **350** (1998), 743–768.
- [V] A. Veca, The Kunze-Stein phenomenon on the isometry group of a tree, *Bull. Austral. Math. Soc.* **65** (2002), 153–174

DARIO CELOTTO: DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO-BICOCCA,  
VIA R. COZZI 53, I-20125 MILANO, ITALY  
D.CELOTTO@CAMPUS.UNIMIB.IT

STEFANO MEDA: DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO-BICOCCA,  
VIA R. COZZI 53, I-20125 MILANO, ITALY  
STEFANO.MEDA@UNIMIB.IT

BLAŻEJ WRÓBEL: MATHEMATICAL INSTITUTE, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, D-53115  
BONN, GERMANY  
& INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCŁAWSKI, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW,  
POLAND  
BLAZEJ.WROBEL@MATH.UNI.WROC.PL